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Estimate of Finite-Q Effect on the Space Growth
Rate of the TM_{11} Unstable Mode in a Linac

by
David C. dePackh
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Electron Beams Branch
Plasma Physics Division

ESTIMATE OF FINITE-Q EFFECT ON THE SPACE GROWTH RATE
OF THE TM_{11} UNSTABLE MODE IN A LINAC

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DAVID C. DEPACKH

In Radiation Project Progress Report No. 4[1] there is an estimate of
the growth rate of the TM_{11} mode due to convective amplification from
transverse electron-beam interaction in a long guide. This estimate was
based on the assumption of zero losses in the guide for the mode in
question (and of zero electrical coupling between adjacent cavities). The
following is an ad hoc calculation of the effect of finite Q while awaiting
the results of a detailed calculation. We continue to assume no electrical
coupling between adjacent cavities. In order to circumvent an elaborate
investigation of the actual geometry, however, we replace each cavity by a
simple resonant LCR circuit and then match coefficients with those derived
in reference 1.

If we imagine a beam passing between the plates of a parallel-plate
vacuum condenser (plate separation = 2a), we can calculate the open-circuit
voltage induced at the plates. It is (in esu) just

$$V_c \cong 2(\partial V/\partial x)x = 4\lambda ex/a$$

for a small displacement x of the beam from the midplane if the linear
density is λ. We now suppose that the circuit contains no loss and that
there is no focusing. The voltage across the capacitor is then
($p \equiv \partial/\partial t = i\omega$, $\omega_0 \equiv (LC)^{-\frac{1}{2}}$)

$$V = (1 + p^2/\omega_0{}^2)^{-1} V_c = (1 - \omega^2/\omega_0{}^2)^{-1} V_c.$$

The corresponding acceleration of an electron is $eV/2a\gamma m$, and the equations of motion are ($\nu \equiv \lambda e^2/mc^2$):

$$(p + vq)x = u$$

$$(p + vq)u = (2c^2\omega_0^2\nu/\gamma a^2)(\omega_0^2 - \omega^2)^{-1} x,$$

where v is the stream velocity, which we henceforth take as c, and $q \equiv \partial/\partial z = -ik$. The spatial growth rate is thus

$$k_i = \left[\left(2\omega_0^2\nu/\gamma a^2\right)\left(\omega_0^2 - \omega^2\right)^{-1}\right]^{\frac{1}{2}}.$$

This can be compared with the value from reference 1:

$$k_i = \left[\left(36.7c^2\nu f/a^4\right)\left(\omega_0^2 - \omega^2\right)^{-1}\right]^{\frac{1}{2}},$$

where ω_0 is the resonant frequency of the cavity in the TM_{10} mode, f is a geometrical factor, and a is the radius. Thus we must compare $2\omega_0^2$ with $36.7fc^2/a^2$. The quantity f is of unit order (though for special parameter choices it can be made zero). For the indicated mode

$$\omega_0^2 = (c^2/a^2)(3.83^2 + \pi^2a^2/L^2)$$

if the cavity length is L. For reasonable cavity lengths the two numbers are therefore of comparable magnitude.

We now proceed with consideration of the LCR resonators, including a nonvanishing resistance and introducing focusing in the form of a betatron frequency. The capacitor voltage then becomes

$$V = (1 + p^2/\omega_0^2 + p/Q\omega_0)^{-1}V_c,$$

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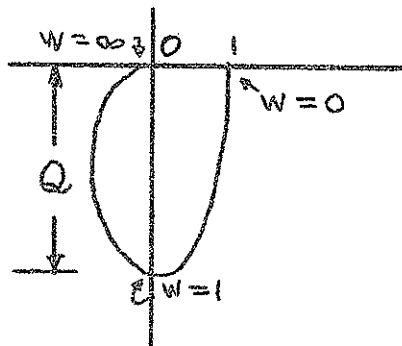where $Q = 1/\omega_o RC$. The equations of motion are now

$$(p + cq)x = u,$$

$$(p + cq)u = \left[-\omega_\beta^2 + (2c^2 \nu/\gamma a^2)(1 + p^2/\omega_o^2 + p/Q\omega_o)^{-1} \right] x.$$
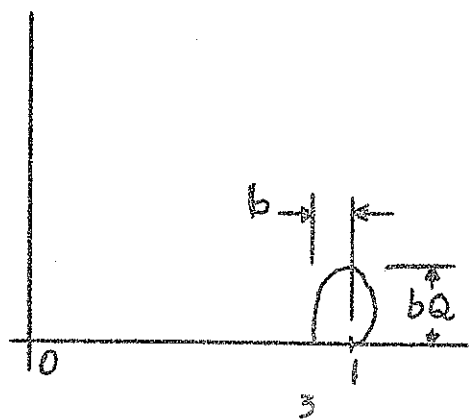
From this one gets

$$ck = \omega \pm \omega_\beta \left[1 - b(1 - w^2 + iw/Q)^{-1} \right]^{\frac{1}{2}},$$

where the definition of the new symbols is obvious. Considering first $(1 - w^2 + iw/Q)^{-1}$, we note that it is zero for $w = \infty$, 1 for $w = 0$, behaves like $1 - iw/Q$ for small w and like $-1/w^2$ for large w, and is $-iQ$ for $w = 1$, so that it looks something like the diagram below:

The corresponding graph of the radicand is of the form:

At maximum growth the radicand is therefore roughly $1 \div ibQ$, and we then have (at $\omega = \omega_0$)

$$ck = \omega_0 \pm \omega_\beta (1 \div \tfrac{1}{2} i b Q)$$

for bQ small. The spatial growth rate is thus

$$k_i = Qcv/\gamma \omega_\beta a^2.$$

If we accept the analogy between the LCR oscillators and the waveguide, we can then write for the latter

$$k_i = 18.4 \, Qfc^3 v/\gamma \omega_\beta \omega_0^2 a^4,$$

provided this is small compared with ω_β/c.

One is inclined to try to reduce the growth rate by making f small. From reference 1 (p. 21)

$$f = (1 \div 0.67 a^2/L^2)^{-1} \left[(2/\pi) \phi (1 - \phi^2/\pi^2)^{-1} \cos \tfrac{1}{2} \phi \right]^2,$$

where $\phi \equiv \omega L/c$. The principal concern is with the resonant frequency; if ω_1 is the TM_{010} frequency, we have approximately $\omega_1 = 2.4 c/a$, giving

$$1 \div 0.67 a^2/L^2 = 0.4 \omega_0^2/\omega_1^2 = 0.4 \phi_0^2/\phi_1^2.$$

Thus near $\omega = \omega_0$

$$f = 2.5 (\phi_1^2/\phi_0^2) \left[(2/\pi) \phi_0 (1 - \phi_0^2/\pi^2)^{-1} \cos \tfrac{1}{2} \phi_0 \right]^2.$$

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The bracket as a function of ϕ_0 can be estimated from the following table:

ϕ_0	[ ]
0	0
$\pi/2$	0.95
π	1.57
2π	1.55
3π	0

We also have

$$\phi_0 = 1.6(\phi_1^2 + 3.9)^{\frac{1}{2}}$$

from the above formulae, so that

$$f = \phi_1^2(\phi_1^2 + 3.9)^{-1}[\ ]^2 .$$

For $0 < \phi_1 < \pi$, f varies about linearly from 0 to 0.74. Practical values of ϕ_1 range from about $\frac{1}{2}\pi$ to π. These give ϕ_0 values between 4.0 and 5.9, corresponding to values of the bracket in the above table near the maximum. This tends to discourage the notion of killing the instability by simply adjusting a/L.

We can estimate the growth rate for $\phi_1 = \pi$, which seems to be the most likely candidate:

$$k_i = 5.2 \times 10^{-12} Q v \omega_1^2 / \gamma \omega_\beta .$$

To get an idea of the actual growth including the increase of γ, we

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suppose that the amplitude is proportional (subscript f = final value) to

$$e^{\int_0^{z_f} k_i \, dz},$$

since this factor accounts for most of the increase, while for k_i we write $K/(\gamma_0 + \gamma'z)$. The integral is then

$$(K/\gamma') \log (\gamma_f/\gamma_0) = (Kz_f/\gamma_0)\left[(\gamma_0/\gamma_f) \log (\gamma_f/\gamma_0)\right].$$

The quantity Kz_f/γ_0 is just the exponent which would be calculated if the initial value of γ_0 were used. It should be observed that it is also the appropriate exponent if ω_β is proportional to $1/\gamma$, which is the case in a constant magnetic focusing field.

If we use $K/\gamma_0 = 0.8\nu$, which corresponds to taking $Q = 10^5$, $\omega_1^2/\omega_\beta = 10^8$, and taking $\gamma_0 = 60$ (out of the first cavity) as a reference, and set $z_f = 3000$ cm, we have on the basis of constant $\omega_\beta \gamma$

$$\nu = 0.0004 \rightarrow 6.8 \text{ amp}$$

for a single e-fold increase. If we suppose ω_β constant, the corresponding value becomes, for $\gamma_f/\gamma_0 = 30$,

$$\nu = 0.0036 \rightarrow 61 \text{ amp}.$$

It therefore seems clear that some way of reducing Q for the unwanted mode, while leaving it unaffected for the TM_{010} mode, is required. It seems reasonable to seek a Q-spoiling factor of the order of 100. This might be accomplished by the use of resistive radial ridges.

References

1. D. C. dePackh, J. B. Ehrman, and T. F. Godlove, "Synchrotron Radiation," Radiation Project Progress Report Number 4, 5 June 1968, p. 17.